When is a trigonometric polynomial not a trigonometric polynomial?

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Abstract

As an application of Bézout's theorem from algebraic geometry, we show that the standard notion of a trigonometric polynomial does not agree with a more naive, but reasonable notion of trigonometric polynomial.

Mathematicians generally accept (see [1,5]) that the term *trigonometric polynomial* refers to a function $f(t) \in C^\infty(\mathbb{R}, \mathbb{C})$ which can be expressed in the following form,

$$f(t) = \sum_{n=0}^{k} a_n \cos(nt) + \sum_{n=1}^{k} b_n \sin(nt)$$

for some nonnegative integer k and complex numbers $a_0, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{C}$. Trigonometric polynomials and their series counterparts, the *Fourier series*, play an important role in many areas of pure and applied mathematics and are likely to be quite familiar to the reader. When reflecting on the terminology, however, it is reasonable to wonder why the term *trigonometric polynomial* is not reserved for a function $g(t)$ of the form,

$$g(t) = \sum_{n=0}^{k} \alpha_n \cos^n(t) + \sum_{n=1}^{k} \beta_n \sin^n(t)$$

for some nonnegative integer k and complex numbers $\alpha_0, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \mathbb{C}$. In this article, we shall refer to these as *naive trigonometric polynomials*.

It is clear that each of these sets of trigonometric polynomials form a subspace of $C^\infty(\mathbb{R}, \mathbb{C})$, but at first glance it isn't obvious that these two subspaces are in fact distinct. An exercise in trigonometric identities, however, shows that any naive trigonometric polynomial can be written as a (standard) trigonometric polynomial. We leave the straightforward details to the reader.

Proposition 1. *Any naive trigonometric polynomial can be written as a (standard) trigonometric polynomial.*

What is curious (and perhaps less clear) is that not all trigonometric polynomials can be written as naive trigonometric polynomials. It is known that $\cos(nt) = T_n(\cos t)$ where T_n is the n-th *Chebyshev polynomial of the first kind*,

and thus $\cos(nt)$ is expressible as a naive trigonometric polynomial. For the sine terms, one has the identity $\sin(nt) = (\sin t) \cdot U_{n-1}(\cos t)$ where U_{n-1} is the $(n-1)$-st *Chebyshev polynomial of the second kind*. Both $T_n(x)$ and $U_n(x)$ are polynomials of degree n which are even or odd functions according as n is even or odd. It follows that $\sin((2k+1)t)$ can be expressed as a naive trigonometric polynomial as well. For the definition and properties of the Chebyshev polynomials, see [4]. Thus, the resolution of our problem is reduced to showing that polynomials of the form $\sin(2kt)$ cannot be expressed as naive trigonometric polynomials. Without too much work, one can provide an elementary argument substantiating this claim and we invite the interested reader to produce one. Our goal here is to prove this fact as an application of the celebrated theorem of Bézout from algebraic geometry. The following version of Bézout's theorem is suitable for our needs and follows as an immediate consequence of the usual statement of Bézout's theorem which involves zero sets of homogeneous polynomials in the complex projective plane $\mathbb{C}P^2$ (see [2, 3, 6]).

Theorem 2 (Bézout's theorem (weak form))**.** *Let $p(x,y)$, $q(x,y) \in \mathbb{C}[x,y]$ be complex polynomials of degree m, n with $\gcd\{p,q\} = 1$. The number of intersection points in $\mathbb{C}^2$ of the two curves $p(x,y) = 0$, $q(x,y) = 0$ is at most mn.*

Proposition 3. *The function $\sin(2kt)$ cannot be represented as a naive trigonometric polynomial.*

Our proof of Proposition 3 uses the following lemma, which states that polynomial relations between $\cos(t)$ and $\sin(t)$ are all consequences of the Pythagorean identity $\cos^2(t) + \sin^2(t) = 1$. For $p(x,y) \in \mathbb{C}[x,y]$, we let $\big(p(x,y)\big)$ denote the ideal generated by $p(x,y)$.

Lemma 4. *Let $R(x,y) \in \mathbb{C}[x,y]$. Then $R\big(\cos(t), \sin(t)\big) = 0$ for every $t \in \mathbb{R}$ if and only if $R(x,y) \in \big(x^2 + y^2 - 1\big)$.*

Proof. If $R(x,y) \in \big(x^2 + y^2 - 1\big)$ then clearly $R\big(\cos(t), \sin(t)\big) \equiv 0$. Conversely, suppose that $R\big(\cos(t), \sin(t)\big)$ is identically zero and suppose that $R(x,y) \notin (x^2+y^2-1)$. Since x^2+y^2-1 is *irreducible*, by Bézout's theorem, it follows that the number of intersection points of the curves $R(x,y) = 0$ and $x^2 + y^2 - 1 = 0$ is at most $2 \cdot \deg(R)$. This implies that $\cos(t)$ and $\sin(t)$ take on only finitely many values as t ranges over all real numbers, an obvious contradiction. □

Remark 5. As pointed out by the referee, a nice alternate proof of lemma 4 using algebraic geometry is to apply Hilbert's *Nullstellensatz* (see, for example, [2,3]). Namely, our hypothesis implies that $R(x,y)$ vanishes on the affine variety defined by the ideal $\big(x^2+y^2-1\big)$. The Nullstellensatz then implies that $R^r \in \big(x^2+y^2-1\big)$ for some natural number r. Thus, $x^2 + y^2 - 1$ divides R^r and irreducibility of $x^2 + y^2 - 1$ further implies that $x^2 + y^2 - 1$ divides $R(x,y)$.

We are now ready to prove Proposition 3.

Proof of Proposition 3. Suppose that there are single-variable complex polynomials P, Q such that $\sin(2kt) = P(\cos(t)) + Q(\sin(t))$ for all $t \in \mathbb{R}$. Using the identity $\sin(2kt) = (\sin t) \cdot U_{2k-1}(\cos t)$ we deduce that $(\sin t) \cdot U_{2k-1}(\cos t) = P(\cos(t)) + Q(\sin(t))$. From Lemma 4, it follows that $U_{2k-1}(x)y - P(x) - Q(y) \in \big(x^2 + y^2 - 1\big)$. That is, there is a polynomial $S(x, y)$ such that

$$U_{2k-1}(x)y - P(x) - Q(y) = S(x, y)(x^2 + y^2 - 1).$$

Let $x^\alpha y^\beta$ be any monomial of S. If $\alpha > 0$ then the right-hand side has a term $x^\alpha y^{2+\beta}$ which is not matched by any term on the left-hand side. So every monomial of S has the form y^β. However, if $\beta > 0$, then the right-side term $x^2 y^\beta$ is not matched on the left side since $U_{2k-1}(x)$ is an odd polynomial. Thus, $S(x, y)$ is reduced to a constant which is clearly impossible. □

Remark 6. Observe that the argument of Lemma 4 can be applied to show that any pair of functions $f(t)$, $g(t)$, each with a range of infinite cardinality, can satisfy essentially at most one polynomial relation. Thus, for example, any polynomial relation of $x = \cosh(t)$ and $y = \sinh(t)$ must be a consequence of $x^2 - y^2 = 1$.

Since we have now shown that the space of naive trigonometric polynomials is a proper subspace of the trigonometric polynomials, we'd like to briefly look at the size of this proper subspace. It turns out that the space of naive trigonometric polynomials is quite small in the following sense. Let $L^2(S^1)$ denote the Hilbert space of square integrable, complex-valued functions on the circle with inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt$. It is well known that $L^2(S^1)$ is the closure of the space of trigonometric polynomials and that every function in $L^2(S^1)$ can be expressed as an L^2-convergent *Fourier series* [5]. Let $\mathcal{N} \subset L^2(S^1)$ be the subspace of all naive trigonometric polynomials. Let the closure of $\mathcal{N}$ be denoted by $\mathrm{cl}(\mathcal{N})$. The next proposition shows that $\mathcal{N}$ has infinite codimension in $L^2(S^1)$.

Proposition 7. *The collection of functions* $\{\sin(2t), \sin(4t), \sin(6t), \ldots\} \subset \mathrm{cl}(\mathcal{N})^\perp$. *In particular, unlike the trigonometric polynomials, the naive trigonometric polynomials are not dense in* $L^2(S^1)$. *Moreover,* $\dim \mathrm{cl}(\mathcal{N})^\perp = \infty$.

Proof. Let k be a positive integer. As mentioned earlier, $\sin(2kt) = U_{2k-1}(\cos t)\sin t$ where $U_{2k-1}(x)$ is the $(2k-1)$-st Chebyshev polynomial of the second kind. We will show that $\sin(2kt)$ is orthogonal to $\mathcal{N}$. Since the collection $\{1, \sin^n(t), \cos^n(t)\}$, for all $n \in \mathbb{N}$, form a spanning set for $\mathcal{N}$, it suffices to show that $\sin(2kt)$ is orthogonal to each function of this spanning set. Since the function $\sin(2kt)\cos^n(t)$ is odd we have

$$\langle \sin(2kt), \cos^n(t)\rangle = \int_{-\pi}^{\pi} \sin(2kt)\cos^n t\, dt = 0.$$

It remains to show that $\langle \sin(2kt), \sin^n(t)\rangle = 0$. Since $U_{2k-1}(x)$ is an odd function of degree $2k-1$, we may write $U_{2k-1}(\cos t) = P(\cos^2 t)\cos t$ for some polynomial

$P(x)$. Thus,

$$\langle \sin(2kt), \sin^n(t) \rangle = \int_{-\pi}^{\pi} \sin(2kt) \sin^n(t)\, dt = \int_{-\pi}^{\pi} U_{2k-1}(\cos(t)) \sin(t) \sin^n(t)\, dt$$

$$= \int_{-\pi}^{\pi} P(\cos^2(t)) \sin^{n+1}(t) \cos(t)\, dt = \int_{-\pi}^{\pi} P(1 - \sin^2(t)) \sin^{n+1}(t) \cos(t)\, dt = 0.$$

The last equality follows from the substitution $w = \sin(t)$. Bilinearity of the inner product implies that $\sin(2kt)$ is orthogonal to the subspace $\mathcal{N}$. Continuity of the inner product implies $\sin(2kt) \in \mathrm{cl}(\mathcal{N})^{\perp}$. Since $\sin(2kt)$ and $\sin(2lt)$ are mutually orthogonal for $k \neq l$, we see that $\mathrm{cl}(\mathcal{N})^{\perp}$ contains the infinite-dimensional subspace spanned by $\{\sin(2t), \sin(4t), \sin(6t), \ldots\}$. □

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